Two approaches to just-in-time scheduling

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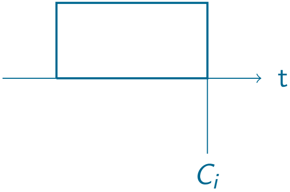
Just-in-time scheduling

Scheduling objectives



completion time: C_i

$$C_{max} = \max_i \{C_i\}$$

$$F = \sum_i (w_i) C_i$$

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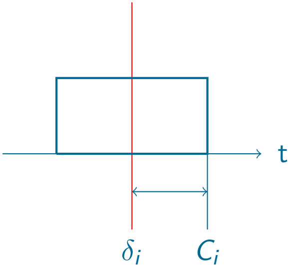


completion time: C_i
due date (deadline): δ_i
lateness: $L_i = C_i - \delta_i$
(weighted) tardiness:
$T_i = \max\{0, C_i - \delta_i\}$

$$C_{max} = \max_i \{C_i\}$$

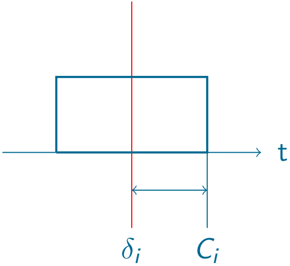
$$F = \sum_i (w_i) C_i$$

$$L_{max} = \max_i \{L_i\}$$

$$T = \sum_i (w_i) T_i$$

Earliness and tardiness costs

Assume the due date δ_i is given for each item i.



$$C_i - \delta_i > 0$$

Item i is completed late.

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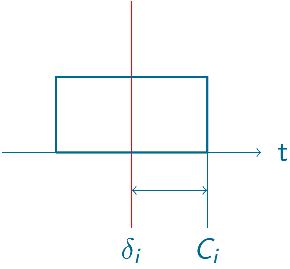

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- financial penalties
- loss of order / client
- loss of reputation

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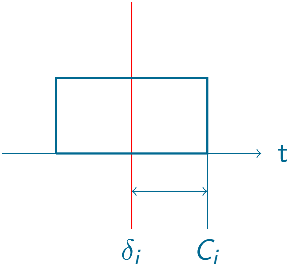

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Non-decreasing function $f_t(t_i)$ where

$$t_i = \begin{cases} C_i - \delta_i & \text{if } C_i > \delta_i \\ 0 & \text{otherwise} \end{cases}$$

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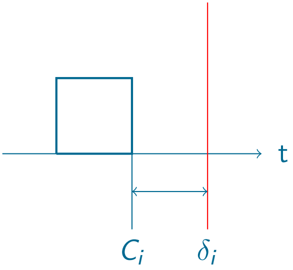

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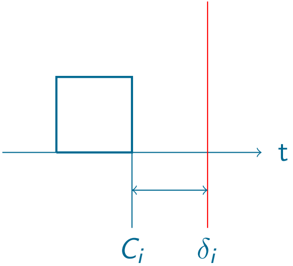

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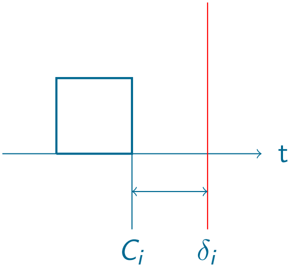

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"Earliness" costs:

- storage space
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Non-decreasing function $f_e(e_i)$ where

$$e_i = \begin{cases} \delta_i - C_i & \text{if } \delta_i > C_i \\ 0 & \text{otherwise} \end{cases}$$

Objective function

$$\text{minimize} \sum_{i=1}^{n} (f_e(e_i) + f_t(t_i))$$

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Some special cases:

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  - scheduling around a common due date $\sum_{i=1}^{n} |C_i - \delta|$ (Kanet 1981, Hall, Kubiak and Sethi 1991, Hoogeveen, van de Velde 1991)

Scheduling with a common due date

minimize $\sum_{i=1}^{n} |\delta - C_i|$

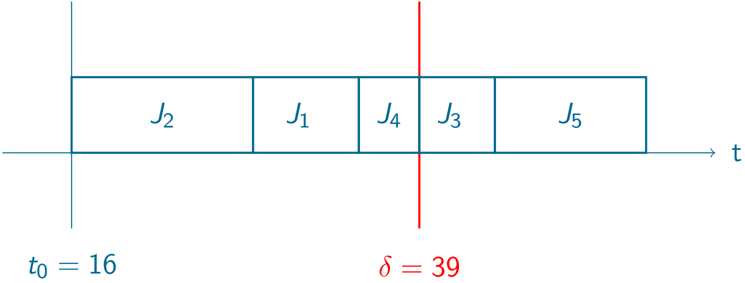
U - set of independent, nonpreemptable jobs, $|U| = n$
p_i - processing time of job i, $i = 1, \ldots, n$
δ - (large) common due date

Algorithm (Kanet, 1981)

```
B ← A ← ∅;
while (U ≠ 0) do
  remove a job k from U such that p_k = max_i{p_i};
  insert job k into the last position in B;
  if (U ≠ ∅) do
    remove a job k from U such that p_k = max_i{p_i};
    insert job k into the first position in A;
  end
end
S ← (B, A);
```

Example

i	1	2	3	4	5
p_i	7	12	5	4	10
$\delta = 39$					



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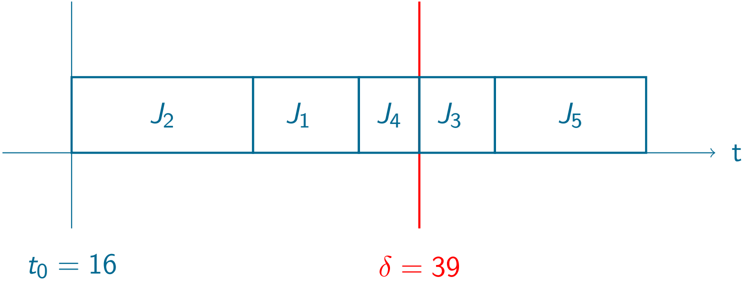

δ "restrictive" $(\delta < 23)$ – NP-hard

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- the longest job is scheduled first,
- V-shaped order of jobs,
- $|B| \geq |A|$,
- $\sum_{i \in B} p_i \geq \sum_{i \in A} p_i$

Computational complexity

Problem	Complexity
$Qm\|\delta_i = \delta\|\sum(\alpha e_i + \beta t_i)$	$O(nlogn)$
$1\|\delta_i = \delta^{res}\|\sum\|C_i - \delta_i\|$	NP-hard
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- non-linear cost functions $\sum_{i=1}^{n}|C_i - \delta|^{\alpha}, \alpha \in \mathbb{R}$

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  - $\alpha \geq 2$ NP-hard (Kubiak 1993)

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  - $\alpha \leq 1$ polynomially solvable (for all concave functions)

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  - $1 < \alpha < 2$ open

Mass production and personalized orders

10%

15%

50%

25%

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Let us assume that assembly of each variant takes the same amount of time (1 unit) and that we need a schedule for 20 time units.

Mass production and personalized orders

	10%	2 pcs.
	15%	3 pcs.
	25%	5 pcs.
	50%	10 pcs.

1 (green)	2 (green)	3 (red)	4 (red)	5 (red)	6 (purple)	7 (purple)	8 (purple)	9 (purple)	10 (purple)	11 (yellow)	12 (yellow)	13 (yellow)	14 (yellow)	15 (yellow)	16 (yellow)	17 (yellow)	18 (yellow)	19 (yellow)	20 (yellow)

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Goal: The number of items completed up to time t as close as possible to amount proportional to product rate.

Mass production and personalized orders

	10%	2 pcs.	$\frac{20}{2} = 10$
	15%	3 pcs.	$\frac{20}{3} = 6\frac{2}{3}$
	25%	5 pcs.	$\frac{20}{4} = 5$
	50%	10 pcs.	$\frac{20}{10} = 2$

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Notation

D	total demand
n	number of product variants
d_i	demand of variant $i, i = 1, \dots, n$
$r_i = \frac{d_i}{D}$	product rate of variant $i, i = 1, \dots, n$
x_{it}	number of items of variant $i, i = 1, \dots, n$ completed up to time $t, t = 1, \dots, D$

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- Ideal number of copies of variant i completed up to time t equals tr_i.
- The goal is to minimize the deviation from this ideal.

Mathematical model

$$\text{minimize} \sum_{i=1}^{n}\sum_{t=1}^{D}|x_{it} - tr_i|$$

$$\text{minimize} \sum_{i=1}^{n}\sum_{t=1}^{D}(x_{it} - tr_i)^2$$

$$\text{minimize} \max_{1\le i\le n}\max_{1\le t\le D}|x_{it} - tr_i|$$

subject to:

$$\sum_{i=1}^{n} x_{it} = t, t = 1, \ldots, D$$

$$0 \le x_{it+1} - x_{it} \le 1, i = 1, \ldots, n; t = 1, \ldots, D$$

$$x_{iD} = d_i, i = 1, \ldots, n$$

$$x_{it} \in \mathcal{N}^+ \cup \{0\}, i = 1, \ldots, n; t = 1, \ldots, D$$

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Kubiak&Sethi: minimize $\sum_{i=1}^{n} \sum_{t=1}^{D} |x_{it} - tr_i|$

Z_j - completion time of the j-th copy of a product

$$\sum_{t=1}^{D} |x_t - tr|$$

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Z_j - completion time of the j-th copy of a product

$$\sum_{t=1}^{D} |x_t - tr| = \sum_{t=1}^{Z_1 - 1} |0 - tr| +$$

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$$\sum_{t=1}^{D} |x_t - tr| = \sum_{t=1}^{Z_1 - 1} |0 - tr| + \sum_{t=Z_1}^{Z_2 - 1} |1 - tr| + \ldots + \sum_{t=Z_d}^{D} |d - tr|$$

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Z_j^* - ideal completion time of the j-th copy of a product.

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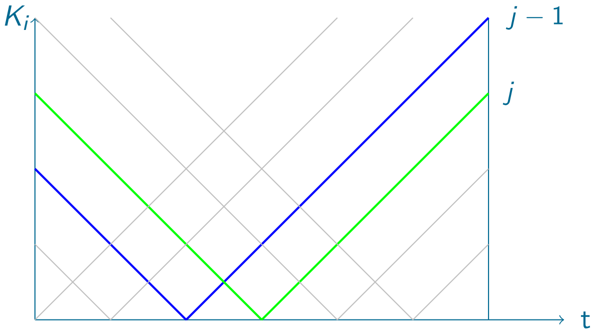
Z_j^{i*} - ideal completion time of the j-th copy of product i.



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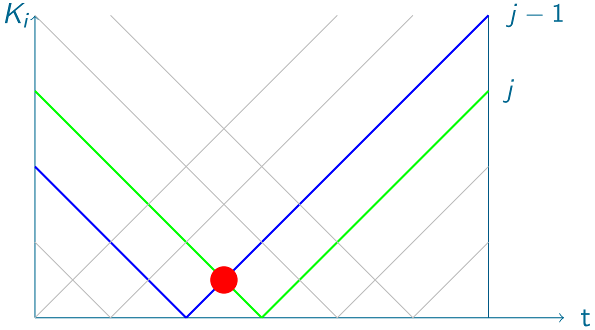

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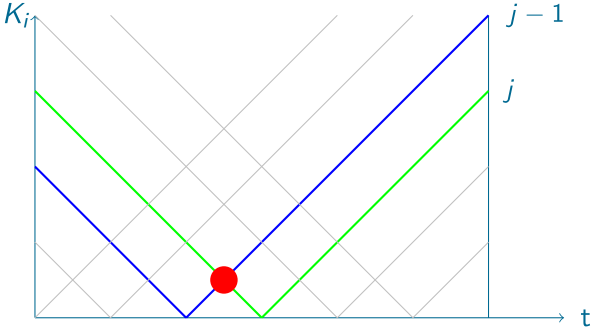

$tr_i - (j-1) = -(tr_i - j)$

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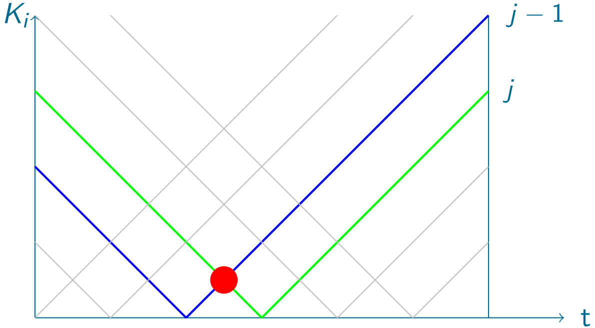

$tr_i - (j-1) = -(tr_i - j)$

$t = \frac{2j-1}{2r_i}$

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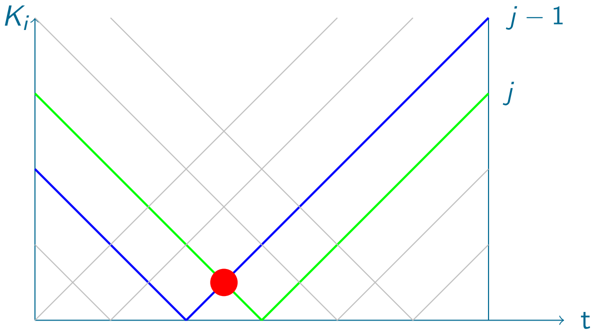

$$Z_j^{i*} = \left\lceil \frac{2j-1}{2r_i} \right\rceil$$

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1. calculate ideal position of (i, j):

$$Z_j^{i*} = \left\lceil \frac{2j - 1}{2r_i} \right\rceil$$

2. calculate the cost C_{jt}^i of scheduling (i, j) in position t:

$$C_{jt}^i = \sum_{l=min(t, Z_j^{i*})}^{max(t, Z_j^{i*})-1} ||lr_i - j| - |lr_i - (j-1)||$$

Kubiak&Sethi: minimize $\sum_{i=1}^{n} \sum_{t=1}^{D} |x_{it} - tr_i|$

Optimal solution is found by solving the following assignment problem:

$$\text{minimize} \sum_{(i,j)\in J} \sum_{t=1}^{D} C^i_{jt} y^i_{jt}$$

$$\text{s.t.} \quad \sum_{t=1}^{D} y^i_{jt} = 1$$

$$\sum_{(i,j)\in J} y^i_{jt} = 1$$

$$(i,j) \in J \Leftrightarrow i \in \{1,2,\ldots,n\} \vee j \in \{1,2,\ldots,d_i\}$$

$$y^i_{jt} = \begin{cases} 1 & \text{if } j\text{-th copy of } i \text{ completes in } t \\ 0 & \text{otherwise} \end{cases}$$

Steiner&Yeomans: minimize $\max_{it} |x_{it} - tr_i|$

Theorem

A just in time sequence with

$$\max_{it} |x_{it} - tr_i| \leq T$$

exists if and only if there exists a sequence that allocates the j-th copy of product i in the interval $[E(i,j), L(i,j)]$ where

$$E(i,j) = \left\lceil \frac{1}{r_i}(j - T) \right\rceil \qquad L(i,j) = \left\lfloor \frac{1}{r_i}(j - 1 + T) + 1 \right\rfloor$$

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Theorem

A just in time sequence with

$$\max_{it} |x_{it} - tr_i| \leq T \leq 1 - \frac{1}{D}$$

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The algorithm tests values $T \in \left\{ \frac{D-d_{max}}{D}, \frac{D-d_{max+1}}{D}, \ldots, \frac{D-1}{D} \right\}$ in ascending order.

Problem of apportionment

Given are:

- the number of states s,
- an integer vector of populations: $\mathbf{p} = (p_1, p_2, p_3, \ldots, p_s)$
- an integer size of the house, $h \geq 0$

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An apportionment of h seats among s states is an integer vector $\mathbf{a}$ such that:

$$\mathbf{a} = (a_1, a_2, a_3, ..., a_s)$$

$$\sum_{i=1}^{s} a_i = h.$$

Goal: Find a fair apportionment.

Problem of apportionment

Given are:

- the number of states s,
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Objective functions - criteria

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Webster, Hamilton:	$\sum_{i=1}^{s} f\left(a_i - \frac{hp_i}{\sum_i p_i}\right)$ where f is any l_p norm
Hill:	$\sum_{i=1}^{s} a_i \left(\frac{p_i}{a_i} - \frac{\sum_i p_i}{h}\right)^2$
Burt & Harris:	$\max_{i,j} \left\{ \frac{p_i}{a_i} - \frac{p_j}{a_j} \right\}$
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Jefferson:	$\max_i \left\{ \frac{a_i}{p_i} \right\}$
Adams:	$\max_i \left\{ \frac{p_i}{a_i} \right\}$

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House monotonicity

Balinski & Young, 1970

House monotone methods

An apportionment method is called house monotone if the number of seats assigned to any state $i, i = 1, \dots, n$, in a parliament of size $h + 1$ is greater than or equal to the number of seats assigned to the same state in a house of size h.

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Property

Hamilton method is not house monotone.

Population monotonicity

Population monotone methods

An apportionment method M is called population monotone if for any two vectors of populations $p, p^{'} > 0$ and vectors of apportionments $a \in (M, h), a' \in (M, h)$ the following implication holds:

$$\frac{p'_{i'}}{p'_{j'}} \geq \frac{p_i}{p_j} \Rightarrow \begin{cases} a'_{i'} \geq a_i \text{ or } a'_{j'} \leq a_j \\ \text{or} \\ \frac{p'_{i'}}{p'_{j'}} = \frac{p_i}{p_j} \text{ and } a'_{i'}, a'_{j'} \text{ can be substituted for } a_i, a_j \text{ in } \mathbf{a} \end{cases}$$

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Property (Balinski & Young)

Any population monotone method is house monotone but not vice versa.

Divisor methods

Divisor methods

Assign the next seat to the state with maximum value of $\frac{p_i}{d(a_i)}$, where $d(a_i)$ is divisor defined below.

Method	Divisor $d(a)$
Adams	a
Dean	$\frac{a(a+1)}{a+0.5}$
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Property (Balinski & Young)

An apportionment method is a divisor method iff it is population monotone.

Quota methods of apportionment

Quota methods

A method of apportionment stays within the quota if all its allocations satisfy the following inequalities:

$$\left\lfloor \frac{h \cdot p_i}{\sum_{i=1}^{s} p_i} \right\rfloor \leq a_i \leq \left\lceil \frac{h \cdot p_i}{\sum_{i=1}^{s} p_i} \right\rceil$$

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Theorem (Balinski & Young)

No method of apportionment exists for $n \geq 4$ and $h \geq n + 3$ that is population monotone and stays within the quota.

Transformation

Scheduling	Apportionment
product $i, i = 1, \ldots, n$	state $i, i = 1, \ldots, s$
d_i demand for product i	p_i population of state i
time unit considered t	size of the house h
x_{it} cumulative number of copies of product i completed up to time t	a_i number of seats assigned to state i in a house of size h

Classification of methods

	HM	NHM
QM	Still, Steiner&Yeomans, quota-divisor methods	Hamilton
NQM	Kubiak&Sethi, divisor methods	

Real time systems (Liu-Layland, 1973)

- n periodic, preemptive and independent tasks
- single processor
- task $i, i = 1, \ldots, n$, is characterized by
  - its request period T_i
  - run-time C_i, such that $T_i \geq C_i$
- execution of the k-th request of task i, which occurs at time $(k-1)T_i$, must finish by the time kT_i

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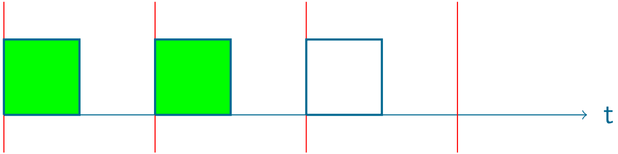

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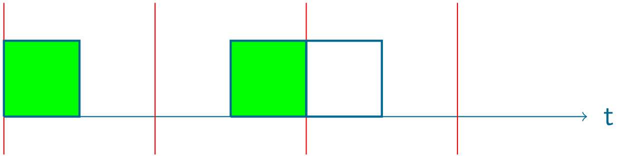

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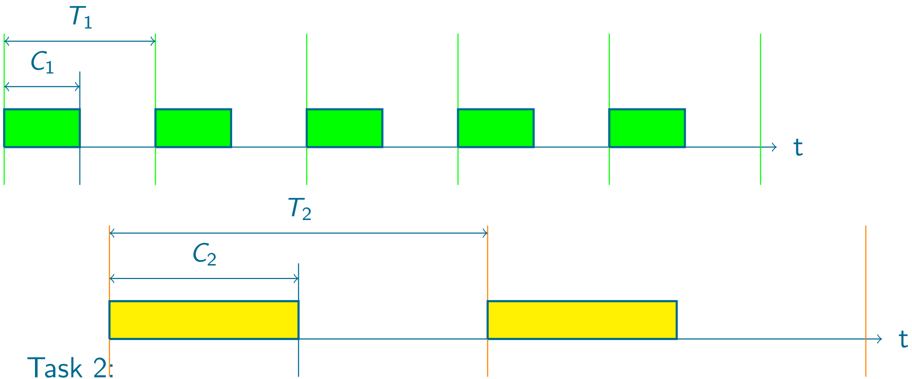

Task 2:

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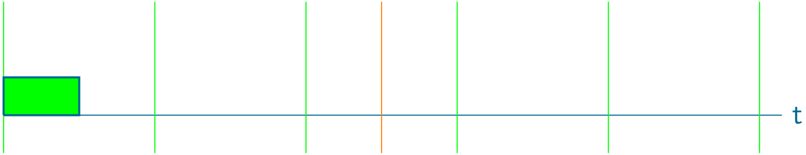
Rate monotonic algorithm: priority to tasks with smaller T_i.



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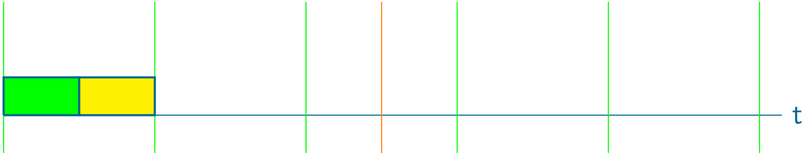

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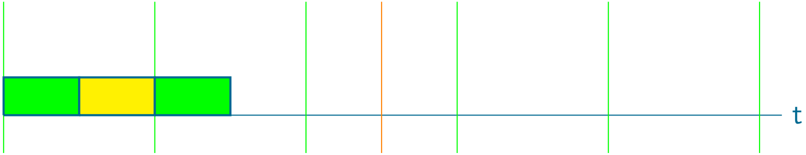

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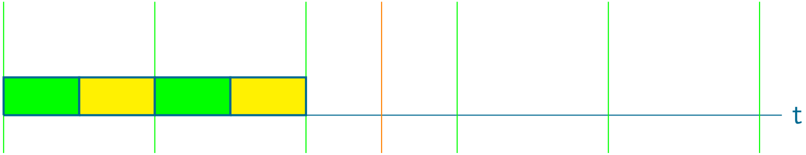

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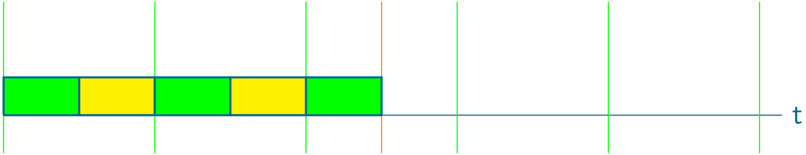

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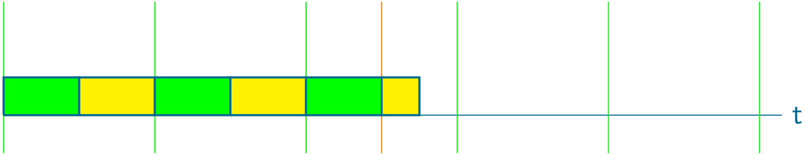

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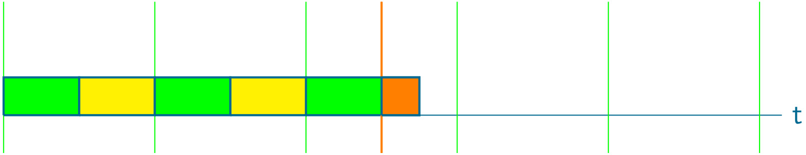

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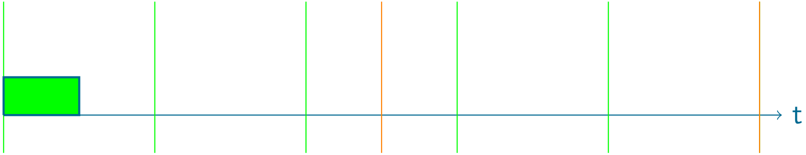

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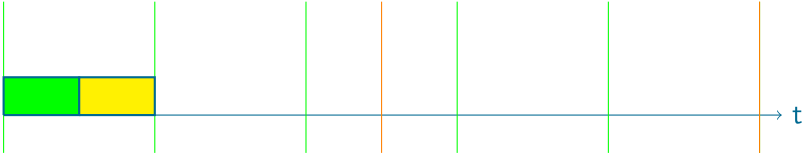

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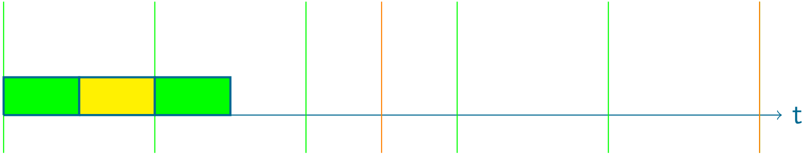

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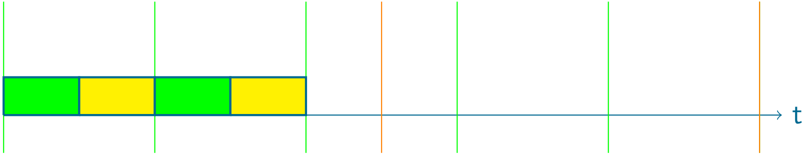

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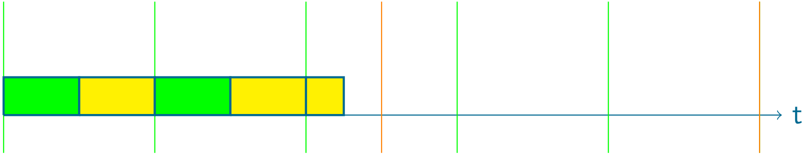

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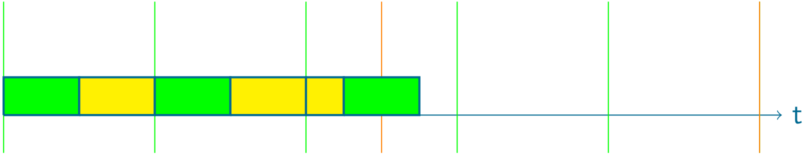

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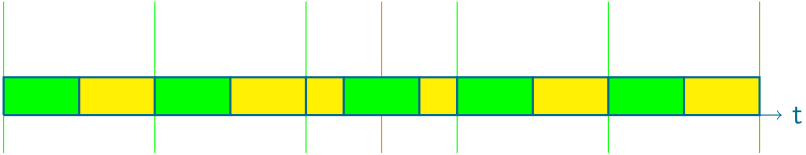

Liu-Layland problem and apportionment

- $\frac{C_i}{T_i}$ expresses the desired proportion of time units allocated to task i in a schedule of any given length - it corresponds to $\frac{p_i}{\sum p_i}$;
- the schedule length corresponds to the size of the house h;

Theorem (Kubiak 2004)

Any house monotone method satisfying the quota solves the Liu-Layland problem.

	HM	NHM
QM	Still, Steiner&Yeomans, quota-divisor methods	Hamilton
NQM	Kubiak&Sethi, divisor methods	

Liu-Layland problem and apportionment

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Theorem (Józefowska et.al. 2008)

Satisfying quota is a necessary condition for any divisor method to solve the Liu–Layland problem.

Corollary

No divisor method solves the Liu–Layland problem.

Stride scheduling

- n competing clients,
- w_i the importance of client i, $i = 1, 2, \ldots, n$,
- goal: allocating units of a discrete resource among clients in such a way that after any allocation the accumulated number of units of the resource possessed by client i is proportional to w_i,

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- dynamic environment: the number of clients n or the values w_i, $i = 1, 2, ..., n$, associated with clients may change in an unpredictable way.

Solution approach (Waldspurger and Weil, 1995)

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- after the quantum passes the process is preempted and the processor can be allocated to another client.

Summary

Stefan Banach (1892-1945)

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A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

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