

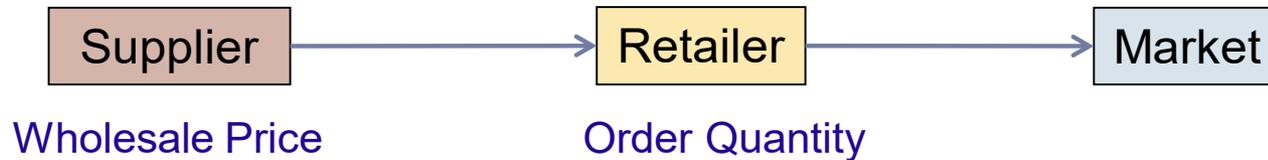
Hierarchical and Mixed Leadership Games for Dynamic Supply Chains: Application to Cost Learning and Co-Operative Advertising

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Conflicts in the Supply Chain Cause Inefficiency

- ▶ The Supplier sells through a Retailer



- ▶ The Supplier sets the wholesale price to maximize his profit.
- ▶ The Retailer, in response, decides on an order quantity to maximize his profit.
- ▶ Their self-serving actions do not maximize the profit of the entire supply chain.
- ▶ This resulting inefficiency is due to **Double Marginalization**. Why?
 1. The Supplier puts up a **margin** over his cost to set the wholesale price.
 2. The Retailer earns a **margin** that equals the difference between the retail price and the wholesale price he pays.
- ▶ This problem is modeled as a Stackelberg Game.

Nash Equilibrium

- ▶ Before we get into the Stackelberg equilibrium, let us discuss the more popular **Nash Equilibrium** by John Nash (1950), whom you might know from the movie *A Beautiful Mind*.



- ▶ In a game of two or more players, a set of decisions by the players is a **Nash Equilibrium** if no player can do better by changing his decision, while the others stay with their decisions.
- ▶ If a player can do better by changing his decision, knowing the decisions of the others and treating them as set in stone, then the set of decisions is not a **Nash Equilibrium**.

Let us now review the bar scene in *A Beautiful Mind*.

The Bar Scene in *A Beautiful Mind*



Nash Equilibrium (cont.)

- ▶ In the movie, Nash thanks the blonde woman for his epiphany, and goes home to write his 26 page thesis, titled *Non-Cooperative Games*, where he develops the concept of the **Nash Equilibrium**.
- ▶ The four friends, I suppose, act on Nash's advice and they all choose the brunette women.
- ▶ Question: Do their decisions form a **Nash Equilibrium**?
- ▶ Answer: No, this is not a **Nash Equilibrium**, since each friend could gain by going for the blonde woman while the others stay with the brunette women.

Nash Equilibrium (cont.)

- ▶ Consider now a simpler version of the scene with two men, **Tom** and **Dick** and two women, Emily and Linda. Both men prefer Emily to Linda and no one gets any if they both choose the same girl.

	Dick	Emily	Linda
Tom			
Emily		(Emily, Emily)	(Emily, Linda)
Linda		(Linda, Emily)	(Linda, Linda)

- ▶ Two Nash Equilibria:
 1. Tom-Emily, Dick-Linda
 2. Tom-Linda, Dick-Emily
- ▶ In (1.), if Tom deviates by going for Linda, he suffers by getting none. Likewise, if Dick deviates by going for Emily, he suffers the same fate.
- ▶ A similar argument holds for (2.).

Stackelberg Equilibrium

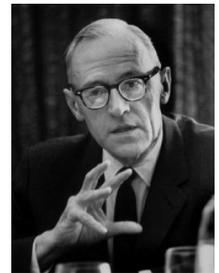
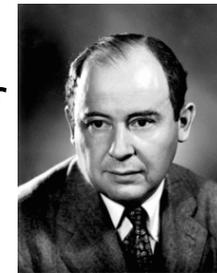
- ▶ Heinrich Freiherr von Stackelberg, *Market Structure and Equilibrium*, 1934.

Leader-Follower games, Stackelberg Games



- ▶ John von Neumann, *Zur Theorie der Gesellschaftsspiele (On the Theory of Parlor Games)*, 1928.
- ▶ John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944.

Zero-Sum games



Stackelberg Equilibrium

- ▶ Let us have Tom as the leader and Dick as the follower.
- ▶ This sequential game is solved by using backward induction. First, we obtain the best response function by Dick for each action by Tom. Then we get Tom's best action.

Response Function - Purple		Response Function - Yellow		Response Function - Blue		Response Function - Grey	
Tom's Decision	Dick's Response	Tom's Decision	Dick's Response	Tom's Decision	Dick's Response	Tom's Decision	Dick's Response
Emily	Linda	Emily	Emily	Emily	Linda	Emily	Emily
Linda	Emily	Linda	Emily	Linda	Linda	Linda	Linda

- ▶ Dick's four possible response functions are shown as Purple, Yellow, Blue and Grey. Clearly, Purple is his best response function.
- ▶ Given Dick's best response function (Purple), it is clear that Tom's best action is to choose Emily. And from Dick's Purple response function, he gets Linda.
- ▶ So the Stackelberg Equilibrium is: Tom-Emily, Dick-Linda.
- ▶ Clearly, if Tom deviates he loses, given the best response function of Dick in Purple. And if Dick changes his response to Yellow, Blue, or Grey, given that Tom has chosen Emily, Dick loses or stays the same.

Differences Between Nash and Stackelberg

- ▶ Obvious difference: Nash involves simultaneous decisions and Stackelberg involves sequential decisions.
- ▶ It is also clear that Tom has the **first-mover advantage** in Stackelberg, as he gets Emily.
- ▶ Subtle difference: Dick's deviation for checking the **Nash Equilibrium** is to change his **action** (going from the woman in the equilibrium to another), whereas his deviation for checking the **Stackelberg Equilibrium** is to change his **response function** (going from **Purple** to another color).
- ▶ There is another, less well-known, difference when it comes to dynamic games. Will discuss it later on Slide 24.



Types of Stackelberg Equilibria in Two-Period Supply Chains

▶ *Open-Loop Solution*

- ▶ The Supplier announces the wholesale prices in periods 1 and 2 at the beginning of the game.
- ▶ The Retailer's best response takes both of these prices to decide on the order quantities in each period.
- ▶ The leader continues to have the *first-mover advantage*.
- ▶ Drawback: The solution does not take into account the realization of the demand in period 1.
- ▶ Drawback: The solution is not *time-consistent*. This means the Supplier has an incentive in period 2 to renege on the wholesale price announced for period 2 at the beginning of the game.
- ▶ For this to work, the Supplier has to commit to his announced prices. It is therefore also called a *Game with Full Commitment*.



Types of Stackelberg Equilibria in Two-Period Supply Chains (cont.)

▶ *Feedback Stackelberg Solution*

- ▶ The Supplier announces only the first-period wholesale price at the beginning of period 1. But then a solution in period 1 must anticipate what would happen in period 2.
- ▶ In period 2, the Supplier announces the second period wholesale price after observing the inventory at the beginning of period 2.
- ▶ The Retailer responds to the Supplier's wholesale price period-wise to obtain his period-wise order quantity.
- ▶ Both take into account the state observed in the second period.
- ▶ The solution is **time-consistent**.
- ▶ Drawback: The leader only has a **period-wise first-mover advantage**.

Dynamic Pricing, Production, and Channel Coordination with Stochastic Learning

Li, Sethi and He, *Production and Operations Management*, 24(6), pp. 857-882, 2015.

- ▶ **Production Cost Learning:** The cost of production in the second period declines in proportion to the quantity produced in the first period.

Production Cost Declines Through Learning

- ▶ Learning-curve (experience curve, Learning-by-doing) phenomenon
 - ▶ Wright 1936: Direct labor cost of manufacturing an airframe fell by 20% with every doubling of cumulative output.
- ▶ Reasons for learning curve effect
 - ▶ Labor efficiency
 - ▶ Less time in discovering manufacturing problems and implementing corrective solutions
 - ▶ Less time in learning, experimenting, or making mistakes
 - ▶ Better use of equipment
 - ▶ Improved production process
 - ▶ Shared experience effects
 - ▶ ...



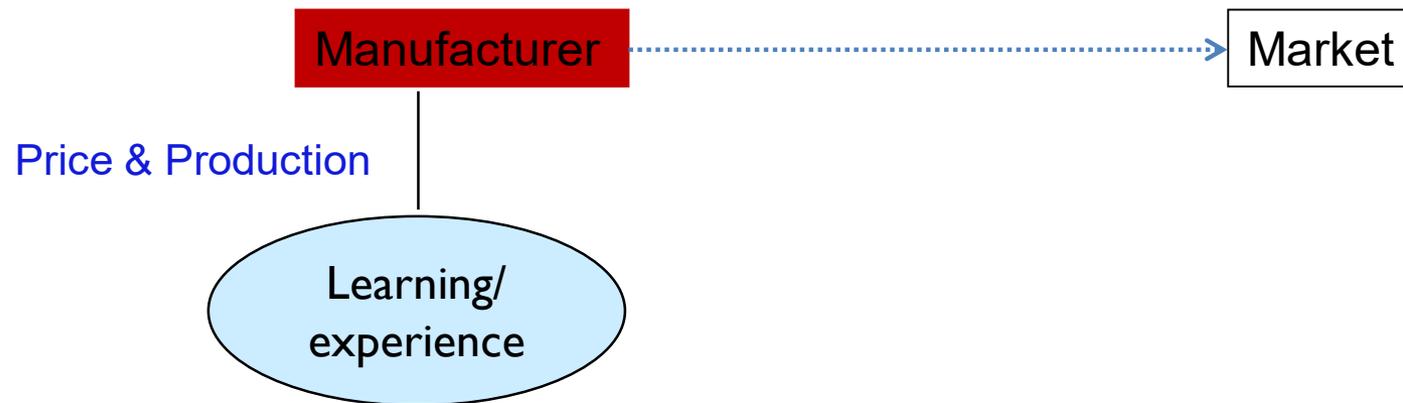
Aircraft Industry

Cost Reduction Process: **Dynamic** and **Random!**

- ▶ Learning effect is **dynamic** and **stochastic**
 - ▶ Cost falls with cumulative production **over time**.
 - ▶ Empirically, the rate at which the observed cost declines tend to **vary** widely, both across and within industries, and even across departments within a firm.
 - ▶ Particularly true for production of a new product, there will be **considerable uncertainty** regarding the rate at which costs will decline with cumulative production.

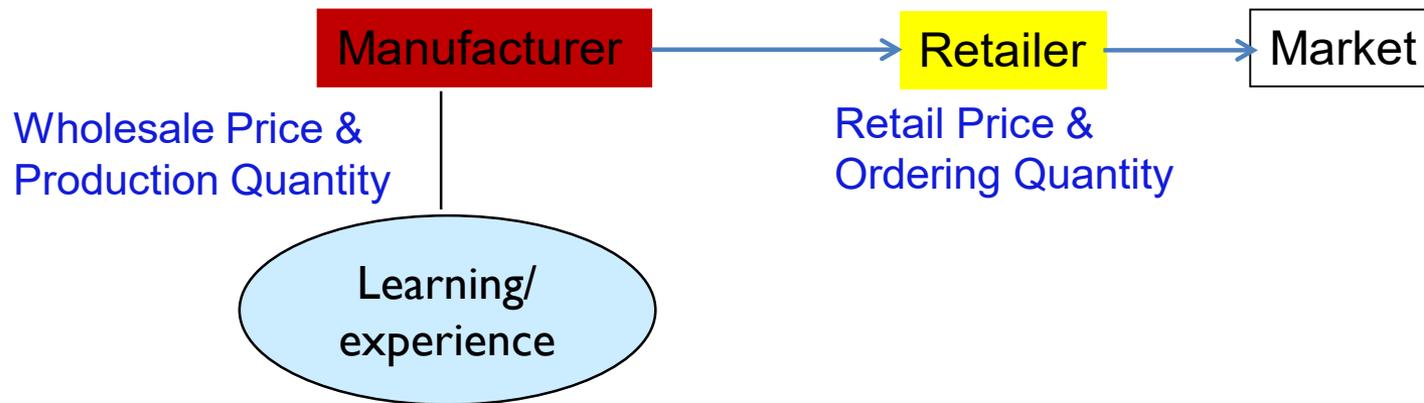
How to Optimize in Presence of Learning?

- ▶ Centralized channel: a Monopolist manufacturer
- ▶ Learning curve creates the interdependency of production cost and cumulative production quantity
 - ▶ Price-dependent demand.
 - ▶ Optimal production quantity? Over produce initially? Optimal price?
- ▶ Strategy recommendation
 - ▶ Reduce price to increase demand resulting in increased production, greater learning and lower future production cost.



More Complicated in a Decentralized Channel

- ▶ Decentralized channel: Manufacturer sells through a retailer
 - ▶ Retailer has pricing power and makes ordering decision.
 - ▶ **Double marginalization**
 - ▶ Higher price and sub-optimal production → *Learning curve effect may not be fully utilized!*
 - ▶ **How to achieve coordination when learning curve is present?**



Outline

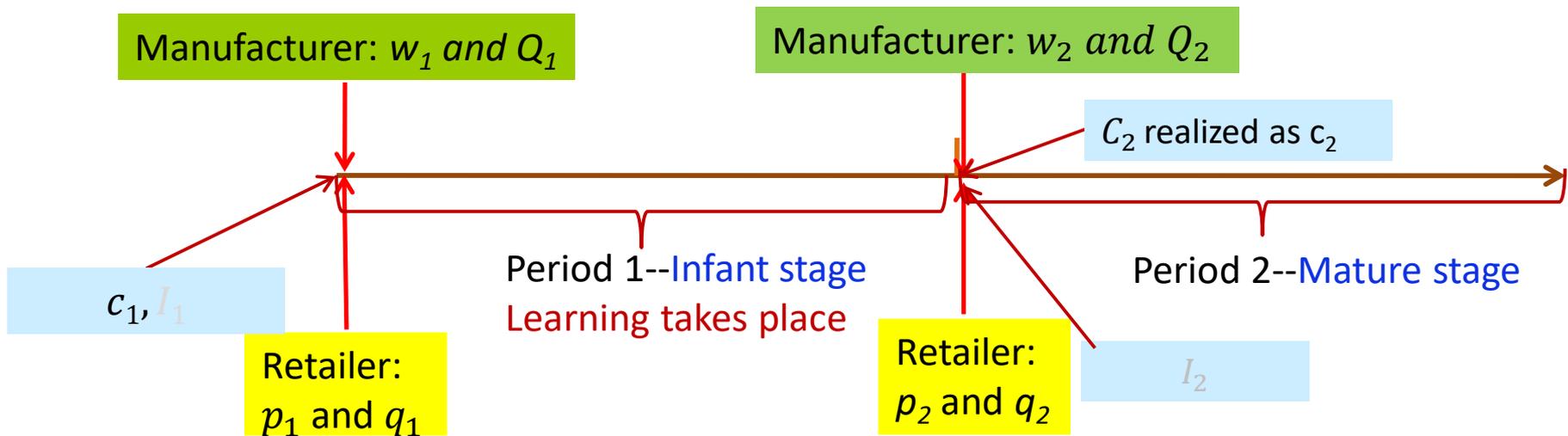
- ▶ Two-period Model Setup
- ▶ Assumptions
- ▶ No Inventory Option
 - ▶ Centralized Channel
 - ▶ Decentralized Channel
 - ▶ Coordination
- ▶ Manufacturer's Inventory Option: Operational Reason
 - ▶ Centralized Channel
 - ▶ Decentralized Channel
 - ▶ Coordination

Notation

- ▶ Two periods, $i = 1, 2$
 - ▶ First period is interpreted as an infant stage; Second period is the mature stage of the industry.
 - ▶ Fudenberg and Tirole (1983), Dasgupta and Stiglitz (1988), Alvarez and Cerda (1999)
- ▶ Demand parameters
 - ▶ a_i : Market potential in Period i
 - ▶ b : Price sensitivity
 - ▶ p_i : Retail price in Period i
 - ▶ w_i : Wholesale price in Period i
- ▶ Cost and learning parameters
 - ▶ c_1 : Production cost in Period 1
 - ▶ C_2 : **Random** production cost in Period 2
 - ▶ c_2 : **Realized** production cost in Period 2
 - ▶ Λ : Random learning rate
- ▶ Production variables
 - ▶ q_i : Retailer's order quantity in Period i
 - ▶ Q_i : Manufacturer's production quantity in Period i
 - ▶ I_i : Inventory at the beginning of Period i

Sequence of Events

- ▶ Manufacturer as leader, retailer as follower
- ▶ Two-period learning curve model
 - Similar two-period models used by Fudenberg and Tirole (1983), Dasgupta and Stiglitz (1988), Alvarez and Cerda (1999)





Assumptions

- ▶ Linear deterministic demand: $D_i = a_i - bp_i, i = 1, 2$
- ▶ Stochastic production cost learning rate: $C_2 = c_1 - \Lambda Q_1$
 - ▶ Random learning rate $\Lambda \in [0, \gamma]$
 - ▶ Mean learning rate μ and standard deviation σ
- ▶ The manufacturer may or may not have the option of inventory carryover.
- ▶ Backorders are not allowed.
- ▶ Manufacturer and retailer are forward-looking.
- ▶ Stackelberg game: manufacturer as leader, retailer as follower.

ASSUMPTION 1 (A1). $a_i - bc_1 > 0, i = 1, 2.$

ASSUMPTION 2 (A2). $c_1 - (a_1 + a_2)\gamma > 0.$

Centralized Channel: No Inventory

▶ Centralized channel's dynamic problem

- ▶ Maximize the expected total profit subject to $C_2 = c_1 - \Lambda Q_1$ with c_1 given.

$$\max_{p_1, Q_1} \left\{ \pi = E \left[p_1 \min\{Q_1, D_1(p_1)\} - c_1 Q_1 + \max_{p_2, Q_2} \{p_2 \min\{Q_2, D_2(p_2)\} - C_2 Q_2\} \right] \right\}$$

- ▶ Optimal policies and profit

$$p_1^*(c_1) = \frac{2(a_1 + bc_1) - b\mu(a_2 - bc_1) - a_1 b^2(\mu^2 + \sigma^2)}{b[4 - b^2(\mu^2 + \sigma^2)]}, \quad Q_1^*(c_1) = a_1 - bp_1^*(c_1),$$

$$p_2^*(c_2) = \frac{a_2 + bc_2}{2b}, \quad Q_2^*(c_2) = \frac{a_2 - bc_2}{2}, \quad C_2^*(c_1) = c_1 - \Lambda Q_1^*(c_1),$$

$$\pi^*(c_1) = \frac{4(a_1 - bc_1)^2 + 4(a_2 - bc_1)^2 + 4b\mu(a_1 - bc_1)(a_2 - bc_1) - b^2\sigma^2(a_2 - bc_1)^2}{4b[4 - b^2(\mu^2 + \sigma^2)]}.$$

Decentralized Channel: No Inventory

- ▶ Manufacturer's and retailer's dynamic problems

$$\Pi_m(c_1, w_1, Q_1, w_2, Q_2, p_1, q_1, p_2, q_2) = E [w_1 q_1 - c_1 Q_1 + w_2 q_2 - C_2 Q_2],$$

$$\Pi_r(c_1, w_1, Q_1, w_2, Q_2, p_1, q_1, p_2, q_2) = E [p_1 \min\{D_1(p_1), q_1\} - w_1 q_1 + p_2 \min\{D_2(p_2), q_2\} - w_2 q_2]$$

Subject to $C_2 = c_1 - \Lambda Q_1$ and c_1 is given.

- ▶ In this setting, the manufacturer is the leader and the retailer is the follower.
 - ▶ In period i ($i = 1, 2$), the manufacturer's decisions are w_i and Q_i , and the retailer's decisions are p_i and q_i .
 - ▶ The production cost c_i is the state variable in period i .

Multi-period Feedback Stackelberg Equilibrium

In a Feedback Stackelberg Equilibrium (FSE), the leader determines her strategy in the feedback form as $u(x, t)$, and the follower's strategy is based on the state as well as the leader's decisions, and is therefore of the form $v(x, t, u(x, t))$, $t = 1, 2, \dots, N$. Notationally, we specify the leader's strategy as $u = (u(x, 1), u(x, 2), \dots, u(x, N))$ and the follower's strategy as $v = (v(x, 1, u(x, 1)), v(x, 2, u(x, 2)), \dots, v(x, N, u(x, N)))$. Let \mathbb{U} and \mathbb{V} denote the spaces of such strategies of the leader and the follower, respectively. Then given $u \in \mathbb{U}$ and $v \in \mathbb{V}$, we denote by $x^{t,y}(i; u, v)$, $i = t, t + 1, \dots, N + 1$, the solution of the state equation

$$\begin{cases} x(i + 1) = x(i) + f(x(i), u(i), v(i), \Lambda(i), i), & i = t, t + 1, \dots, N, \\ x(t) = y, \end{cases}$$

$$\begin{aligned} J_L^{t,y}(u, v) &= J_L^{t,y}(u(\cdot), v(\cdot, u(\cdot))) \\ &= E \left[\sum_{i=t}^N \pi_L(x^{t,y}(i; u, v), u(x^{t,y}(i; u, v), i), v(x^{t,y}(i; u, v), i, u(x^{t,y}(i; u, v), i)), \Lambda(i)) + S_L(x^{t,y}(N + 1; u, v)) \right], \\ J_F^{t,y}(u, v) &= J_F^{t,y}(u(\cdot), v(\cdot, u(\cdot))) \\ &= E \left[\sum_{i=t}^N \pi_F(x^{t,y}(i; u, v), u(x^{t,y}(i; u, v), i), v(x^{t,y}(i; u, v), i, u(x^{t,y}(i; u, v), i)), \Lambda(i)) + S_F(x^{t,y}(N + 1; u, v)) \right], \end{aligned}$$

Multi-period Feedback Stackelberg Equilibrium

Definition A pair of strategies $u^* \in \mathbb{U}$ and $v^* \in \mathbb{V}$ is called a feedback Stackelberg equilibrium if the following holds:

$$J_L^{t,y}(u^*(\cdot), v^*(\cdot, u^*(\cdot))) \geq J_L^{t,y}(u(\cdot), v^*(\cdot, u(\cdot))), \forall u \in \mathbb{U}, \forall y \in \mathbb{R}^n, \forall t \in [1, N]$$

$$J_F^{t,y}(u^*(\cdot), v^*(\cdot, u^*(\cdot))) \geq J_F^{t,y}(u^*(\cdot), v(\cdot, u^*(\cdot))), \forall v \in \mathbb{V}, \forall y \in \mathbb{R}^n, \forall t \in [1, N].$$

- ▶ u^* can be replaced by u in the second inequality above for single-period problems and for open-loop solutions in multi-period problems.
- ▶ In this setting,

$$\Pi_m(c_1, w_1, Q_1, w_2, Q_2, p_1, q_1, p_2, q_2) = J_L^{1,c_1}(u, v),$$

$$\Pi_r(c_1, w_1, Q_1, w_2, Q_2, p_1, q_1, p_2, q_2) = J_F^{1,c_1}(u, v).$$

Decentralized Channel: No Inventory

Table 1: Decisions and Policies in the Decentralized Channel

	Without Inventory Option
Wholesale price $w_1^*(c_1)$	$\frac{8(a_1+bc_1)-2b\mu(a_2-bc_1)-a_1b^2(\mu^2+\sigma^2)}{b(16-b^2(\mu^2+\sigma^2))}$
Production Quantity $Q_1^*(c_1)$	$\frac{4(a_1-bc_1)+b\mu(a_2-bc_1)}{16-b^2(\mu^2+\sigma^2)}$
Order quantity $q_1^*(c_1, w_1, Q_1)$	$\min\{(a_1 - bw_1)/2, Q_1\}$
Retail price $p_1^*(c_1, w_1, Q_1)$	$\max\{(a_1 + bw_1)/2b, (a_1 - Q_1)/b\}$
Second-period cost $C_2^*(c_1)$	$c_1 - \Lambda Q_1^*(c_1)$
Wholesale price $w_2^*(c_2)$	$(a_2 + bc_2)/2b$
Production quantity $Q_2^*(c_2)$	$(a_2 - bc_2)/4$
Order quantity $q_2^*(c_2, w_2, Q_2)$	$\min\{(a_2 - bw_2)/2, Q_2\}$
Retail price $p_2^*(c_2, w_2, Q_2)$	$\max\{(a_2 + bw_2)/2b, (a_2 - Q_2)/b\}$



Revenue Sharing Contract

- ▶ Given the solution in the centralized and decentralized settings, it can be seen that the supply chain profit in the decentralized setting is lower than that in the centralized setting.
- ▶ In order to achieve the centralized profit, coordinating contracts such as Revenue Sharing or Buy Back Contracts are proposed in the literature.
- ▶ Our focus here is to develop a Revenue Sharing Contract in our two-period problem.
- ▶ The manufacturer proposes a revenue sharing contract with a set of *contingent* wholesale prices and revenue sharing rates $\{w_1(c_1), \phi_1, w_2(c_2), \phi_2\}$, $0 \leq \phi_1, \phi_2 \leq 1$.
 - ▶ ϕ_i : Retailer's portion of revenue in period i .

Revenue Sharing: No Inventory

- ▶ Superscript “ rs ”: Quantity under revenue sharing contract
- ▶ The retailer’s problem is

$$\max_{0 \leq p_1 \leq a_1/b, 0 \leq q_1 \leq Q_1} E[\phi_1 p_1 \min\{q_1, D_1(p_1)\} - w_1(c_1)q_1 + \phi_2 p_2 \min\{q_2, D_2(p_2)\} - w_2(C_2)q_2]$$

subject to $C_2 = c_1 - \Lambda Q_1$ and c_1 is given.

- ▶ Coordinating contract

Proposition 6. For any $\phi_1, \phi_2 \in [0, 1]$, if the manufacturer sets $w_1^{rs}(c_1) = \frac{\phi_1(4c_1 - 2\mu(a_2 - bc_1) - a_1b(\mu^2 + \sigma^2))}{4 - b^2(\mu^2 + \sigma^2)}$

and $w_2^{rs}(c_2) = \phi_2 c_2$, then $\{w_1^{rs}(c_1), \phi_1, w_2^{rs}(c_2), \phi_2\}$ is a coordinating contract.

- ▶ Implications: Manufacturer and retailer can flexibly negotiate over revenue sharing rates in two periods, and agree on a contract that eliminates double marginalization.

Co-op Advertising with Two Competing Retailers: A Feedback Stackelberg-Nash Game

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Appears in the Conference Proceedings

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An Application: Cooperative Advertising

- Co-op advertising
 - An arrangement whereby a manufacturer pays for some or all of the costs of the **local advertising** undertaken by the retailer for the manufacturer's product



- **Participation rate:** Specifies the percentage of advertising that the manufacturer will share with the retailer.

Coop Advertising: Some empirical findings

- Total channel expenditure on co-op advertising in 2000 was estimated at \$15 billion, compared to \$5 billion in 1987. Estimates for 2007 were \$25 billion (Nagler 2006)
- The average participation rate is 75% (Dutta et al. 1995)
- 25–40% of retailers' local promotion expenditures (including advertising) are funded by manufacturers (Dant & Berger 1996)

Industrial products (machinery, construction equipment, hardware)	69%
Consumer products	74%
Consumer convenience products (e.g., books, milk, bread, toothpaste, health aids)	88%
Consumer non-convenience products (e.g., women's apparel, tires, furniture)	68%

Coop Advertising

Recent Surveys:

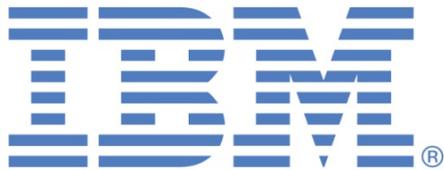
Jorgensen S. & Zaccour A. (2014). A survey of game-theoretic models of cooperative advertising. *European Journal of Operational Research*.

Aust G. & Buscher U. (2014). Cooperative advertising models in supply chain management: A review. *European Journal of Operational Research*.

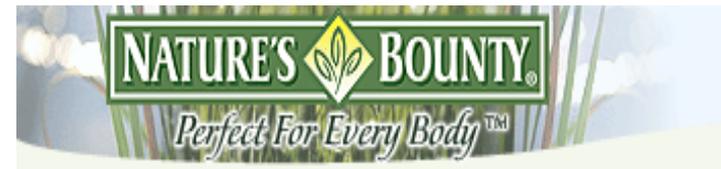
A one manufacturer one retailer Stackelberg Model:

He X., Prasad A. & Sethi S. (2009). Cooperative Advertising and Pricing in a Dynamic Stochastic Supply Chain: Feedback Stackelberg Strategies. *Production and Operations Management*.

IBM: 50% of advertising cost



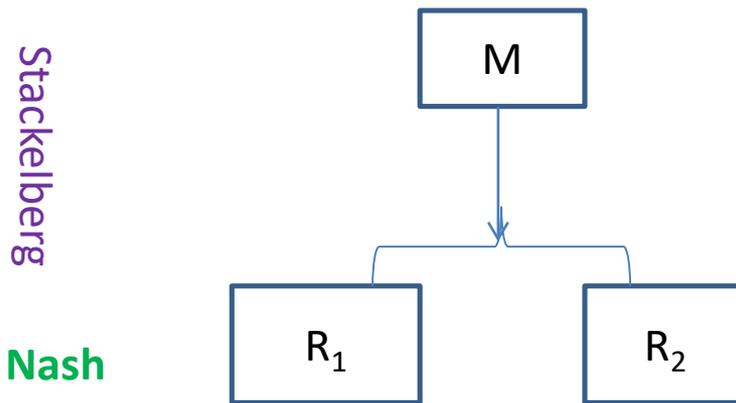
Nature's Bounty: 50%



Apple: 75% of the media costs

But what should be the participation rate?

Co-op Advertising with Two Competing Retailers



- The manufacturer (M) announces his participation rates $\theta_1(t)$ and $\theta_2(t)$ for the local advertising of retailers R_1 and R_2 , respectively.
- In response, R_1 and R_2 choose their advertising efforts
- Start at time t : $x(t)$ is market share of R_1 and $1 - x(t)$ is that of R_2

Problem Formulation: Stackelberg-Nash Game

- Market share dynamics based on the Sethi (1983) model*

$$\begin{aligned}\dot{x}(t) &= \rho_1 u_1(t) \sqrt{1-x(t)} - \rho_2 u_2(t) \sqrt{x(t)} - \delta_1 x(t) + \delta_2 (1-x(t)), \\ x(0) &= x_0 \in [0,1]\end{aligned}$$

- Each player advertises to inform the buyers of the other player and each player loses some of its buyers to the other player.

- Objective function of Retailer 1 (R_1)

$$\pi_1 = \int_0^{\infty} e^{-rt} (m_1 x(t) - (1 - \theta_1(t)) u_1^2(t)) dt$$

- Objective function of Retailer 2 (R_2)

$$\pi_2 = \int_0^{\infty} e^{-rt} (m_2 (1 - x(t)) - (1 - \theta_2(t)) u_2^2(t)) dt$$

- Objective function of the Manufacturer (M)

$$\pi = \max_{\theta_1(t), \theta_2(t), t \geq 0} \int_0^{\infty} e^{-rt} [(M_1 x(t) + M_2 (1 - x(t)) - \theta_1(t) (u_1(t))^2 - \theta_2(t) (u_2(t))^2] dt$$

* Source: https://en.wikipedia.org/wiki/Sethi_model

Feedback Stackelberg-Nash Equilibria in Mixed Leadership Games with an Application to Cooperative Advertising

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Joint work with Alain Bensoussan, Shaokuan Chen, Anshuman Chutani,
Chi Chung Siu, and Phillip Yam

Outline

- Motivation
- Literature on Solutions for Stackelberg Differential Games
- Stochastic Feedback Stackelberg-Nash games
- A mixed leadership cooperative advertising model

Motivation

- Static Stackelberg game
 - One manufacturer one retailer
 - Double marginalization
 - Contract theory

Research Items:

1. What is the equilibrium concept for a multi-period or differential game with both upstream and downstream competition?
2. Find an equilibrium solution of a mixed leadership game of cooperative advertising.

Literature on Solutions of Stackelberg Differential Games

- Closed-loop memory-less information structure
 - Deterministic case: Papavassilopoulos and Cruz (1979)
 - Stochastic: Bensoussan et al. (2015)
- Feedback information structure
 - Discrete-time case: Simaan and Cruz (1973)
 - Backward induction procedure to find the solution
 - The leader has stage-wise advantage
 - Continuous-time case: Basar and Haurie (1984)
 - Feedback Stackelberg equilibrium based on a discretized version of the original game
 - Limit of the time-discretized game remains open; suggests instantaneous advantage for the leader
 - Continuous-time stochastic case: Bensoussan et al. (2019)
 - Feedback Stackelberg equilibrium based on a static Stackelberg game at the Hamiltonian level
 - Sufficient condition for a feedback Stackelberg equilibrium

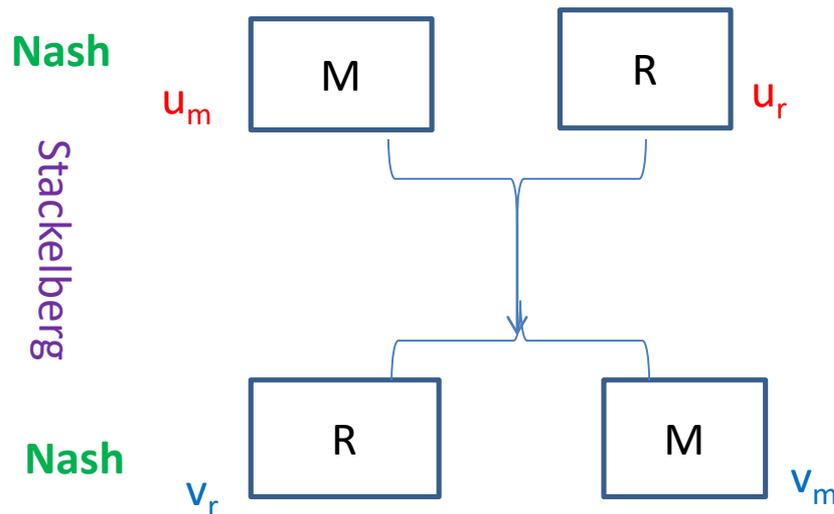
Stochastic Feedback Stackelberg-Nash Games

- Extend deterministic games (Basar and Haurie, 1984) to the stochastic case
 - Static Stackelberg game at Hamiltonian level + dynamic programming
 - Unify feedback Nash and feedback Stackelberg solutions

New Perspective: Mixed leadership game

- Feature: Each player has both **leading** and **following** strategy
- Equilibrium concept
 - Feedback **Stackelberg-Nash** solution
- Application: Cooperative Advertising between a Manufacturer and a Retailer

A variation of cooperative advertising (mixed leadership game)



- State at time t (market share): $x(t)$
- The manufacturer (Player 1) announces his participation rate (u_m) for the retailer's local advertising level and the retailer (Player 2) announces his participation rate (u_r) for the manufacturer's national advertising level
- In response both the parties choose their advertising efforts, i.e., v_m and v_r

Problem Formulation

- Market share dynamics based on the Sethi (1983) model*

$$dx = \left((av_r + bv_m)\sqrt{1-x} - \delta x \right) dt + \sigma(x)dW(t), \quad x(0) = x_0 \in [0,1]$$

- Objective function of the manufacturer

$$J_m = \mathbb{E} \int_0^{\infty} e^{-\rho t} (Mx - u_m v_r^2 - (1 - u_r) v_m^2) dt$$

- Objective function of the retailer

$$J_r = \mathbb{E} \int_0^{\infty} e^{-\rho t} (Rx - (1 - u_m) v_r^2 - u_r v_m^2) dt$$

* Source: https://en.wikipedia.org/wiki/Sethi_model

Stackelberg-Nash games

- Hamiltonians for the manufacturer and the retailer

$$H_m(x, u_m, u_r, v_m, v_r, p_m) = Mx - u_m v_r^2 - (1 - u_r) v_m^2 + p_m \left((a v_r + b v_m) \sqrt{1 - x} - \delta x \right)$$

$$H_r(x, u_m, u_r, v_m, v_r, p_r) = Rx - (1 - u_m) v_r^2 - u_r v_m^2 + p_r \left((a v_r + b v_m) \sqrt{1 - x} - \delta x \right)$$

- **Leading** decisions: participation rates (u_m, u_r)
- **Following** decisions: advertising rates (v_m, v_r)
- Nash game for advertising rates (v_m, v_r)

$$\begin{cases} -2(1 - u_r) v_m + p_m b \sqrt{1 - x} = 0 \\ -2(1 - u_m) v_r + p_r a \sqrt{1 - x} = 0 \end{cases} \Rightarrow v_m = \frac{p_m b \sqrt{1 - x}}{2(1 - u_r)}, \quad v_r = \frac{p_r a \sqrt{1 - x}}{2(1 - u_m)}$$

Stackelberg-Nash games

- Nash game for participation rates

$$H_m = Mx - u_m \frac{p_r^2 a^2 (1-x)}{4(1-u_m)^2} - (1-u_r) \frac{p_m^2 b^2 (1-x)}{4(1-u_r)^2} + p_m \left(\frac{p_r a^2 (1-x)}{2(1-u_m)} + \frac{p_m b^2 (1-x)}{2(1-u_r)} - \delta x \right)$$

$$H_r = Rx - (1-u_m) \frac{p_r^2 a^2 (1-x)}{4(1-u_m)^2} - u_r \frac{p_m^2 b^2 (1-x)}{4(1-u_r)^2} + p_r \left(\frac{p_r a^2 (1-x)}{2(1-u_m)} + \frac{p_m b^2 (1-x)}{2(1-u_r)} - \delta x \right)$$



$$\begin{cases}
 u_m = \operatorname{argmax} H_m = \begin{cases} 0, & \text{if } 2p_m \leq p_r, \\ \frac{2p_m - p_r}{2p_m + p_r}, & \text{otherwise} \end{cases} \\
 u_r = \operatorname{argmax} H_r = \begin{cases} 0, & \text{if } 2p_r \leq p_m, \\ \frac{2p_r - p_m}{2p_r + p_m}, & \text{otherwise} \end{cases}
 \end{cases}$$

Verification Theorem

- Solve the system

$$\begin{cases} -\rho V_r(x) + \frac{1}{2}\sigma^2(x) V_r''(x) + H_r(x, u_m, u_r, v_m, v_r, V_r'(x)) = 0, \\ -\rho V_m(x) + \frac{1}{2}\sigma^2(x) V_m''(x) + H_m(x, u_m, u_r, v_m, v_r, V_m'(x)) = 0, \end{cases}$$



$$u_m = \begin{cases} 0, & \text{if } 2 V_m'(x) \leq V_r'(x), \\ \frac{2 V_m'(x) - V_r'(x)}{2 V_m'(x) + V_r'(x)}, & \text{otherwise} \end{cases}$$

$$u_r = \begin{cases} 0, & \text{if } 2 V_r'(x) \leq V_m'(x), \\ \frac{2 V_r'(x) - V_m'(x)}{2 V_r'(x) + V_m'(x)}, & \text{otherwise} \end{cases}$$

$$v_m = \frac{V_m'(x)b\sqrt{1-x}}{2(1-u_r)},$$

$$v_r = \frac{V_r'(x)a\sqrt{1-x}}{2(1-u_m)}$$

An equilibrium

- We see if linear value functions give an equilibrium solution

$$V_r(x) = \alpha_r + \beta_r x, \quad V_m(x) = \alpha_m + \beta_m x,$$

- We have **three** different cases:

1. $\beta_m \leq \frac{\beta_r}{2}$  $u_m = 0, u_r = \frac{2\beta_r - \beta_m}{2\beta_r + \beta_m} \geq 0$

$$-\rho\alpha_r + \frac{4\beta_r^2 a^2 + (2\beta_r + \beta_m)^2 b^2}{16} = 0,$$

$$-\rho\beta_r + R - \frac{4\beta_r^2 a^2 + (2\beta_r + \beta_m)^2 b^2}{16} - \beta_r \delta = 0,$$

$$-\rho\alpha_m + \frac{4\beta_m\beta_r a^2 + \beta_m(\beta_m + 2\beta_r)b^2}{8} = 0,$$

$$-\rho\beta_m + M - \frac{4\beta_m\beta_r a^2 + \beta_m(\beta_m + 2\beta_r)b^2}{8} - \beta_m \delta = 0$$

An equilibrium

$$2. \quad \frac{\beta_m}{2} < \beta_r < 2\beta_m \quad \longrightarrow \quad u_m = \frac{2\beta_m - \beta_r}{2\beta_m + \beta_r} > 0, \quad u_r = \frac{2\beta_r - \beta_m}{2\beta_r + \beta_m} > 0$$

$$-\rho\alpha_r + \frac{2\beta_r (2\beta_m + \beta_r)a^2 + (2\beta_r + \beta_m)^2 b^2}{16} = 0,$$

$$-\rho\beta_r + R - \frac{2\beta_r (2\beta_m + \beta_r)a^2 + (2\beta_r + \beta_m)^2 b^2}{16} - \beta_r\delta = 0,$$

$$-\rho\alpha_m + \frac{(2\beta_m + \beta_r)^2 a^2 + 2\beta_m (2\beta_r + \beta_m) b^2}{16} = 0,$$

$$-\rho\beta_m + M - \frac{(2\beta_m + \beta_r)^2 a^2 + 2\beta_m (2\beta_r + \beta_m) b^2}{16} - \beta_m\delta = 0$$

An equilibrium

$$3. \beta_m \geq 2\beta_r \quad \longrightarrow \quad u_m = \frac{2\beta_m - \beta_r}{2\beta_m + \beta_r} \geq 0, \quad u_r = 0$$

$$-\rho\alpha_r + \frac{\beta_r (2\beta_m + \beta_r)a^2 + 4\beta_r\beta_m b^2}{8} = 0,$$

$$-\rho\beta_r + R - \frac{\beta_r (2\beta_m + \beta_r)a^2 + 4\beta_r\beta_m b^2}{8} - \beta_r\delta = 0,$$

$$-\rho\alpha_m + \frac{(2\beta_m + \beta_r)^2 a^2 + 4\beta_m^2 b^2}{16} = 0,$$

$$-\rho\beta_m + M - \frac{(2\beta_m + \beta_r)^2 a^2 + 4\beta_m^2 b^2}{16} - \beta_m\delta = 0$$

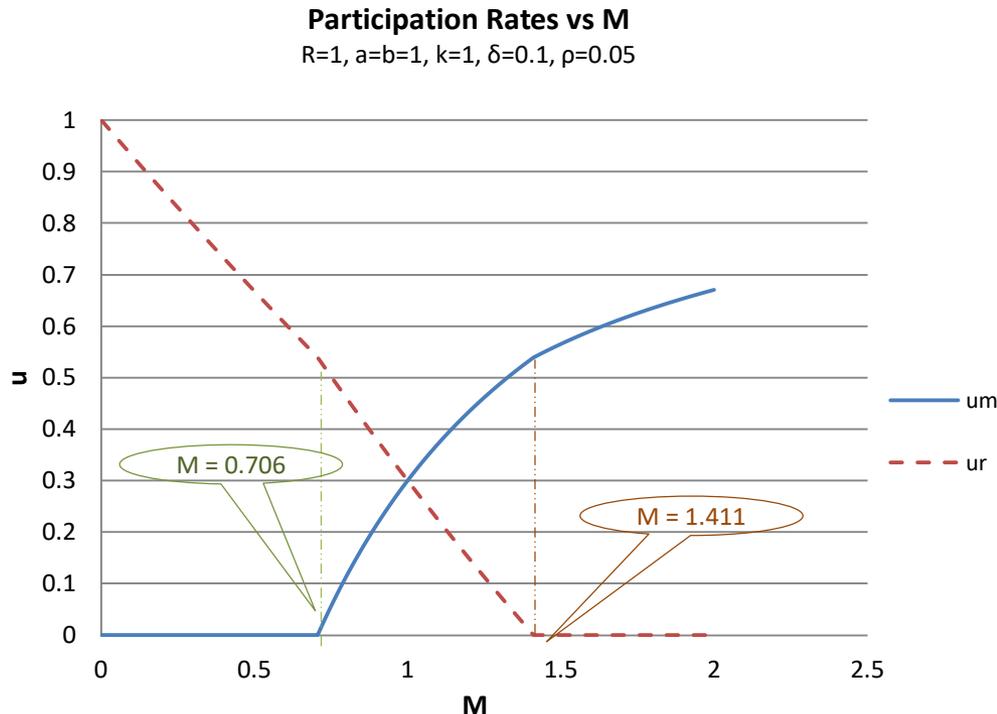
Numerical Analysis

- Extension: A more generalized state dynamic (with synergy, $k \geq 0$)

$$dx = \left((av_r + bv_m + k\sqrt{v_r v_m})\sqrt{1-x} - \delta x \right) dt + \sigma(x)dW(t), \quad x(0) = x_0 \in [0,1]$$

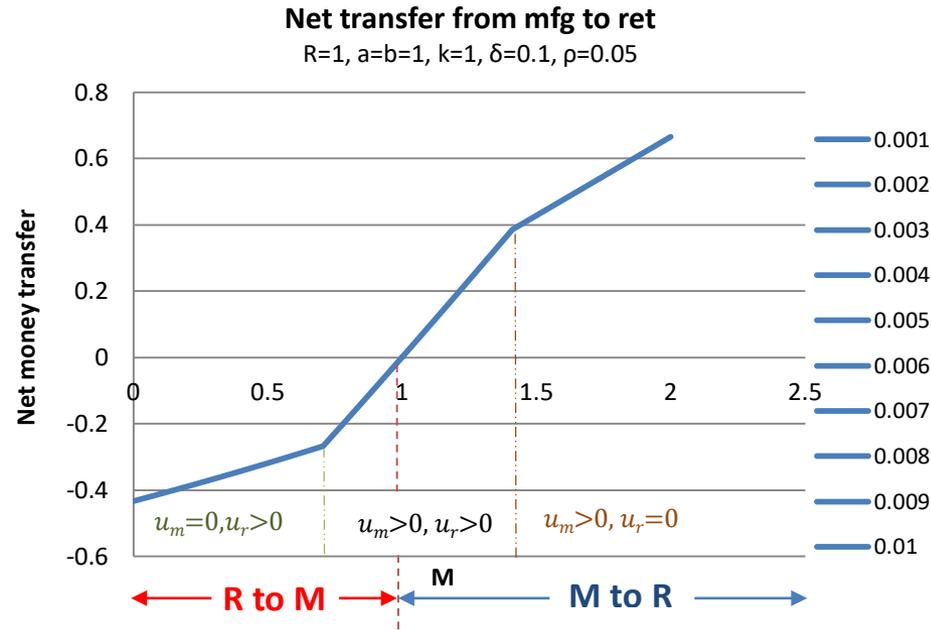
- Some insights on the impact of various model parameters

Impact of Manufacturer's Margin (M) on participation rates



- As $M \uparrow$ manufacturer has a greater incentive to advertise more and therefore his support to retailer (u_m) \uparrow .
- As $M \uparrow$, retailer knows manufacturer has a higher incentive for advertising himself, and so his support to manufacturer (u_r) \downarrow .
 - Beyond a threshold, when $M \gg R$, the retailer's support $u_r = 0$.
- The results are opposite when $M \downarrow$, i.e., when $R \uparrow$ relative to M .
- When $R \ll M$, the manufacturer's support $u_r = 0$.

Impact of M on net transfer from Mfr to Ret



The graph shows net transfer from Mfr to Ret per unit of uncaptured market share, i.e.,

$$\frac{(mfr's\ contribution\ for\ local - ret's\ contribution\ for\ national)}{(1-x)}$$

As $M \uparrow$, manufacturer's incentive \uparrow , and net transfer from manufacturer to retailer \uparrow

$M > R \Rightarrow$ net transfer is from mfr to ret, and it is the other way around when $M < R$

Conclusion

- ▶ Difference between Nash and Stackelberg Games.
- ▶ Types of Stackelberg Equilibria in Two-Period Supply Chains.
- ▶ Definition of Multi-Period Feedback Stackelberg Equilibrium.
- ▶ Dynamic Pricing, Production in a Supply Chain with Cost Learning.
- ▶ Revenue Sharing Contracts that Coordinate the Two-Period Supply Chain.
- ▶ Co-op Advertising with Two Competing Retailers: A Feedback Stackelberg-Nash Game.
- ▶ A Mixed Leadership Game of Cooperative Advertising: A Feedback Stackelberg-Nash Game.

References

- He , X., Prasad, A., Sethi, S.P., and Gutierrez, G.J. (2007) A Survey of Stackelberg Differential Game Models in Supply and Marketing Channels, *J. Syst. Sci. Syst. Eng.* 16(4): 385-413.
- He , X., Prasad, A., Sethi, S.P. (2009) Cooperative Advertising and Pricing in a Dynamic Stochastic Supply Chain: Feedback Stackelberg Strategies, *Production and Operations Management* 18(1): 78-94.
- Basar , T., Bensoussan, A., and Sethi, S.P. (2010) Differential Games with Mixed Leadership: The Open-Loop Solution, *Applied Mathematics and Computation* 217(3): 972-979.
- Bensoussan et al. (2013) Linear Quadratic Differential Games with Mixed Leadership: The Open-Loop Solution, *NACO* 3(1): 95-108.
- Basar , T., and Olsder, G.J. (1982) *Dynamic Noncooperative Game Theory*, SIAM Classics in Applied Mathematics 23.
- Bensoussan, A., Chen, S., and Sethi, S.P., The Maximum Principle for Global Solutions of Stochastic Stackelberg Differential Games, *SIAM Journal on Control and Optimization*, 53(4), 2015, 1956–1981.
- Bensoussan, A., Chen, S., Chutani, A., and Sethi, S.P., Feedback Stackelberg-Nash Equilibrium in Mixed Leadership Games with an Application to Cooperative Advertising., *SIAM Journal on Control and Optimization* , 57(5), 2019, 3413-3444.

Thank you!

If you have any questions,
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